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Skew orthogonal polynomials and the partly symmetric real Ginibre ensemble

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Abstract
The partly symmetric real Ginibre ensemble consists of matrices formed as linear combinations of real symmetric and real anti-symmetric Gaussian random matrices. Such matrices typically have both real and complex eigenvalues. For a fixed number of real eigenvalues, an earlier work has given the explicit form of the joint eigenvalue probability density function. We use this to derive a Pfaffian formula for the corresponding summed up generalized partition function. This Pfaffian formula allows the probability that there are exactly k eigenvalues to be written as a determinant with explicit entries. It can be used too to give the explicit form of the correlation functions, provided certain skew orthogonal polynomials are computed. This task is accomplished in terms of Hermite polynomials, and allows us to proceed to analyze various scaling limits of the correlations, including that in which the matrices are only weakly non-symmetric.

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1. Introduction

In random matrix theory the Ginibre ensembles [14] refer to Gaussian random matrices with either real, complex or real quaternion entries, which are all independent. In distinction to ensembles of Hermitian matrices, the support of the eigenvalues in the Ginibre ensembles is a disk in the complex plane. Thus, the eigenvalue distribution can be regarded as specifying a point process in a two-dimensional domain. In this paper, we will study the point process associated with the eigenvalue distribution for random matrices which interpolate between the real Ginibre ensemble and the Gaussian orthogonal ensemble (GOE) of real symmetric matrices.

The two-dimensional point process associated with the eigenvalue distribution for random matrices which interpolate between the complex Ginibre ensemble and the Gaussian unitary ensemble (GUE) of complex Hermitian matrices has been the subject of an earlier study

[13]. Similarly, the eigenvalue distribution for random matrices interpolating between the real quaternion Ginibre ensemble and the Gaussian symplectic ensemble (GSE) of real quaternion Hermitian matrices has also been analyzed as a point process [17]. The interpolating ensemble to be studied herein is thus the last of those naturally associated with the Ginibre ensembles to be considered from this viewpoint. Such studies are well motivated for their relevance to Efetov's theory of directional quantum chaos [8], in which a special role is played by the interpolating ensembles in the weak non-Hermiticity limit.

The point processes associated with the Ginibre ensembles have the feature of being proportional to $e^{-\beta U}$ for potentials U which are the sum of one- and two-body terms, the two-body terms being logarithmic. As the logarithmic pair potential is that for two-dimensional charges, there is thus an analogy with the equilibrium statistical mechanics of certain two-dimensional one-component Coulomb systems (see e.g. [9] and references therein). This is most immediate in the case of the complex Ginibre ensemble, for which the eigenvalue probability density function (PDF) is proportional to $e^{-\beta U}$ with $\beta = 2$ and

$$U = \frac{1}{2}\sum_{l=1}^{N}|z_l|^2 - \sum_{1\leqslant j<k\leqslant N}\log|z_k - z_j|, \qquad z_j := x_j + iy_j. \tag{1.1}$$

Potential (1.1) is due to N unit two-dimensional charges, repelling via the logarithmic pair potential $-\log|z - z'|$, and with a smeared out disk of uniform neutralizing charge centred about the origin of charge density $-1/\pi$, which creates the one-body harmonic potential $\frac{1}{2}|z|^2$. Note that the disk must have radius $\sqrt{N}$ to neutralize the N mobile charges. As we expect equilibrium Coulomb systems to be locally charge neutral (otherwise an electric field would be created, and the system would go out of equilibrium), the particle density should to leading order also be a disk centred about the origin of radius $\sqrt{N}$, a fact which can be checked upon exact calculation of the one-point correlation. We remark that recently variants on the eigenvalue problem for complex Ginibre matrices have been formulated [19], which have analogies with one-component Coulomb systems on the surface of a sphere [4] and in a hyperbolic disk [16] (in relation to the latter see too the work [21] on certain random complex polynomials).

The eigenvalue PDF for matrices interpolating between the complex Ginibre ensemble and the Gaussian unitary ensemble also has a Coulomb gas analogy. Such matrices can be written in the form $H + i v A$, where H and A are Hermitian matrices from the Gaussian unitary ensemble, scaled so that the joint PDF of the elements is proportional to $\exp\left(-\frac{1}{1+\tau}\,\mathrm{Tr}X^2\right)$, $\tau = (1-v^2)/(1+v^2)$. The eigenvalue PDF can be computed as being proportional to [13]

$$\exp\left(-\frac{1}{1-\tau^2}\sum_{j=1}^{N}\left(|z_j|^2 - \frac{\tau}{2}\left(z_j^2 + \bar{z}_j^2\right)\right)\right)\prod_{1\leqslant j<k\leqslant N}|z_k - z_j|^2. \tag{1.2}$$

Here the one-body potential can be interpreted as being due to a uniformly charged ellipse, charge density $-1/\pi(1-\tau^2)$, semi-axes A and B given by [10, 12]

$$A = \sqrt{N}(1+\tau), \qquad B = \sqrt{N}(1-\tau). \tag{1.3}$$

Again, one can anticipate that the particle density will to leading order be of this same shape, a fact which can be verified by exact computation of the one-point correlation [12].

As for the complex Ginibre ensemble, the leading eigenvalue support of the real Ginibre ensemble is also a disk [6]. Moreover, in the case of matrices interpolating between the real Ginibre ensemble and the GOE, it has been anticipated that the support will be an ellipse [20]. By exact calculation of the one-point function, verification of this fact is given in section 5.1 below.

We begin in section 2 by defining matrices interpolating between the real Ginibre ensemble and the GOE in terms of a linear combination of random real symmetric and anti-symmetric matrices. Knowledge of the eigenvalue PDF for the real Ginibre ensemble allows the eigenvalue PDF of such matrices to be computed exactly. The resulting functional form is of an identical structure to that for the real Ginibre ensemble, allowing in particular the corresponding generalized partition function to be written as a Pfaffian. This is considered in section 3 and used to give the probability $p_{k,N}$ that an $N \times N$ (N even) member of the interpolating ensemble has exactly k real eigenvalues as a determinant of size $N/2$. The Pfaffian formula for the generalized partition function implies the k-point correlation functions can be written as a $2k \times 2k$ Pfaffian, with entries given in terms of skew orthogonal polynomials. The latter are with respect to the skew inner product corresponding to the entries of the Pfaffian. The main technical task is the computation of these skew orthogonal polynomials, which is undertaken in section 4. In section 5 simple expressions in terms of Hermite polynomials are obtained. The final section, section 6, is concerned with various scaled limits of the real–real and complex–complex correlations. For fixed τ the forms obtained are identical to the $\tau = 0$ case (real Ginibre ensemble) except for a simple scaling of the coordinates which accounts for the change in the two-dimensional density. The case $\tau = 0$ has been previously studied in [11, 25] and most comprehensively in the work of Borodin and Sinclair [3] (this latter work treats too the general real–complex correlations). The weakly non-symmetric limit of the correlations is also computed.

2. Definition of the ensemble and the eigenvalue PDF

Let S be an element of the Gaussian orthogonal ensemble of $N \times N$ real symmetric matrices, and thus have PDF of its independent elements proportional to $\mathrm{e}^{-\mathrm{Tr}S^2/2}$ (equivalently, the diagonal elements have distribution $N[0, 1]$ while the strictly upper triangular elements have distribution $N[0, 1/\sqrt{2}]$). Let A be an element of the anti-symmetric Gaussian orthogonal ensemble of real anti-symmetric matrices which has PDF of its independent elements proportional to $\mathrm{e}^{\mathrm{Tr}A^2/2}$ (each strictly upper triangular element is thus independently distributed according to $N[0, 1/\sqrt{2}]$). With $0 < \tau < 1$ and $c := (1-\tau)/(1+\tau)$ define random matrices X according to

$$X = \frac{1}{\sqrt{b}}(S + \sqrt{c}A). \tag{2.1}$$

When $\tau = 0$ and $b = 1$, $X = S + A$. In this case each element of X is independently distributed as a standard Gaussian $N[0, 1]$ and so X is a member of the real Ginibre ensemble. When $\tau = 1$ and $b = 1$, $X = S$ and so X is a member of the GOE. Thus, X interpolates between the real Ginibre ensemble and the GOE as the parameter τ is varied from 1 down to 0.

The probability measure associated with the matrices S and A is

$$(2\pi)^{-N/2}\pi^{-N(N-1)/2}\,\mathrm{e}^{-(\mathrm{Tr}S^2-\mathrm{Tr}A^2)/2}(\mathrm{d}A)(\mathrm{d}S). \tag{2.2}$$

From (2.1) we compute that

$$(\mathrm{d}X) = 2^{N(N-1)/2}(\sqrt{c})^{N(N-1)/2}(\sqrt{b})^{-N^2}(\mathrm{d}S)(\mathrm{d}A),$$

and we too observe that S and A can be written in terms of X and X^T. Thus we can change variables in (2.2) to obtain for the PDF of the matrices X

$$A_{\tau,b}\exp\left(-\frac{b}{2(1-\tau)}(\mathrm{Tr}\,XX^T - \tau\mathrm{Tr}\,X^2)\right), \tag{2.3}$$

where

$$A_{\tau,b} = (\sqrt{c})^{-N(N-1)/2}(\sqrt{b})^{N^2}(2\pi)^{-N^2/2}. \tag{2.4}$$

We seek the eigenvalue PDF corresponding to (2.3).

A fundamental point is that because X is real, there is a non-zero probability that the eigenvalue will be real, and furthermore all complex eigenvalues must occur in complex conjugate pairs. Thus the eigenvalue PDF decomposes into a sum of PDFs $P_{k,(N-k)/2}(\{\lambda_j\}_{j=1,\dots,k}; \{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2}; \tau, b)$ corresponding to having k real eigenvalues $\{\lambda_j\}_{j=1,\dots,k}$ and $(N-k)/2$ complex conjugate pairs of eigenvalues $\{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2}$ (for this to be non-zero k and N must have the same parity, a condition which will henceforth be assumed). In the case $\tau = 0$ and $b = 1$ the probability $P_{k,(N-k)/2}$ has been computed explicitly in [6, 20, 23] to give

$$\begin{aligned} &P_{k,(N-k)/2}(\{\lambda_j\}_{j=1,\dots,k}; \{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2}; 0, 1) \\ &\quad = \frac{1}{2^{N(N+1)/4}\prod_{l=1}^{N}\Gamma(l/2)}\frac{2^{(N-k)/2}}{k!((N-k)/2)!} \\ &\qquad \times \left|\Delta(\{\lambda_l\}_{l=1,\dots,k} \cup \{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2})\right| \\ &\qquad \times \mathrm{e}^{-\sum_{j=1}^{k}\lambda_j^2/2}\,\mathrm{e}^{\sum_{j=1}^{(N-k)/2}(y_j^2-x_j^2)}\prod_{j=1}^{(N-k)/2}\operatorname{erfc}(\sqrt{2}y_j), \end{aligned} \tag{2.5}$$

where $\Delta(\{z_p\}_{p=1,\dots,m}) := \prod_{j<l}^{m}(z_l - z_j)$. In fact the eigenvalue PDF for general τ and b is closely related to this functional form.

In the case $\tau = 0$ and $b = 1$ we read off from (2.3) that the PDF of the elements of X is given by $A_{0,1}\,\mathrm{e}^{-\mathrm{Tr}XX^T/2}$. With $X \mapsto \sqrt{b}X/(1-\tau)^{1/2}$ this latter PDF becomes

$$A_{0,1}b^{N^2/2}(1-\tau)^{-N^2/2}\,\mathrm{e}^{-b\,\mathrm{Tr}XX^T/2(1-\tau)}, \tag{2.6}$$

while the eigenvalue PDF is obtained from (2.5) by a simple scaling and so is equal to

$$\begin{aligned} &b^{N/2}(1-\tau)^{-N/2}P_{k,(N-k)/2}(\{\sqrt{b}\lambda_j/(1-\tau)^{1/2}\}_{j=1,\dots,k}; \\ &\{\sqrt{b}x_j/(1-\tau)^{1/2} \pm \mathrm{i}\sqrt{b}y_j/(1-\tau)^{1/2}\}_{j=1,\dots,(N-k)/2}; 0, 1). \end{aligned}$$

Now (2.6) is a factor in (2.3) while the remaining factor proportional to $\exp\left(\frac{\tau b}{2(1-\tau)}\,\mathrm{Tr}X^2\right)$ can immediately be written in terms of the eigenvalues of X. It follows from these considerations that [20]

$$\begin{aligned} &P_{k,(N-k)/2}(\{\lambda_j\}_{j=1,\dots,k}; \{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2}; \tau, b) \\ &\quad = \frac{A_{\tau,b}}{A_{0,1}}(1-\tau)^{N(N-1)/2}\exp\left(\frac{\tau b}{2(1-\tau)}\left(\sum_{j=1}^{k}\lambda_j^2 + 2\sum_{j=1}^{(N-k)/2}\left(x_j^2 - y_j^2\right)\right)\right)P_{k,(N-k)/2} \\ &(\{\sqrt{b}\lambda_j/(1-\tau)^{1/2}\}_{j=1,\dots,k}; \{\sqrt{b}x_j/(1-\tau)^{1/2} \pm \mathrm{i}\sqrt{b}y_j/(1-\tau)^{1/2}\}_{j=1,\dots,(N-k)/2}; 0, 1) \\ &\quad = \frac{(\sqrt{b})^{N(N+1)/2}(\sqrt{1+\tau})^{N(N-1)/2}}{2^{N(N+1)/4}\prod_{l=1}^{N}\Gamma(l/2)}\frac{2^{(N-k)/2}}{k!((N-k)/2)!} \\ &\qquad \times |\Delta(\{\lambda_l\}_{l=1,\dots,k} \cup \{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2})| \\ &\qquad \times \mathrm{e}^{-b\sum_{j=1}^{k}\lambda_j^2/2}\,\mathrm{e}^{b\sum_{j=1}^{(N-k)/2}(y_j^2-x_j^2)}\prod_{j=1}^{(N-k)/2}\operatorname{erfc}\left(\sqrt{\frac{2b}{1-\tau}}y_j\right). \end{aligned} \tag{2.7}$$

Integrating $P_{k,(N-k)/2}$ over $\lambda_j \in \mathbb{R} (j = 1, \dots, k)$ and $(x_j, y_j) \in \mathbb{R}_2^+ (j = 1, \dots, (N-k)/2)$, where $\mathbb{R}_2^+ := \{(x, y) \in \mathbb{R}^2 : y > 0\}$ gives the probability $p_{k,N}$ say that a matrix of form (2.1) has exactly k real eigenvalues. The parameter b then scales out of the problem and so for convenience may be set equal to unity. A discussion of a systematic approach to the calculation of these probabilities is given in section 3 below. The case $k = N$, when all eigenvalues are real, is special and can be considered immediately. Thus, comparing (2.5) and (2.7) one sees that

$$P_{N,0}(\{\lambda_j\}_{j=1,\dots,N};\tau,1) = (\sqrt{b})^{N^2/2}(\sqrt{1+\tau})^{N(N-1)/2}P_{N,0}(\{\lambda_j\}_{j=1,\dots,N};0,1).$$

But we know from [6] that $p_{N,N}\big|_{\substack{\tau=0\\ b=1}} = 2^{-N(N-1)/4}$ and so for general $0 \leqslant \tau \leqslant 1$

$$p_{N,N} = \left(\frac{2}{1+\tau}\right)^{-N(N-1)/4}. \tag{2.8}$$

3. Generalized partition function and the probabilities $p_{k,N}$

The generalized partition function associated with PDF (2.7) is defined by

$$Z_{k,(N-k)/2}[u,v] = \int_{-\infty}^{\infty} \mathrm{d}\lambda_1 \cdots \int_{-\infty}^{\infty} \mathrm{d}\lambda_k \prod_{l=1}^{k} u(\lambda_l) \int_{\mathbb{R}_+^2} \mathrm{d}x_1\,\mathrm{d}y_1 \cdots \int_{\mathbb{R}_+^2} \mathrm{d}x_{(N-k)/2}\,\mathrm{d}y_{(N-k)/2}$$
$$\times \prod_{l=1}^{(N-k)/2} v(x_l,y_l) P_{k,N-k}(\{\lambda_j\}_{j=1,\dots,k};\{x_j \pm \mathrm{i}y_j\}_{j=1,\dots,(N-k)/2};\tau,b). \tag{3.1}$$

In view of the functional form (2.7) this is structurally identical to the case $\tau = 0$ and $b = 1$, when a Pfaffian formula is known [24]. Consequently, (3.1) too has a Pfaffian evaluation.

Proposition 1. *Let* $\{p_{l-1}(x)\}_{l=1,\dots,N}$ *be a set of monic polynomials of the indexed degree, and let*

$$\alpha_{j,k}[u] = \int_{-\infty}^{\infty} \mathrm{d}x\, u(x) \int_{-\infty}^{\infty} \mathrm{d}y\, u(y)\, \mathrm{e}^{-b(x^2+y^2)/2} p_{j-1}(x) p_{k-1}(y)\, \mathrm{sgn}(y-x) \tag{3.2}$$

$$\beta_{j,k}[v] = 2\mathrm{i} \int_{\mathbb{R}_+^2} \mathrm{d}x\,\mathrm{d}y\, v(x,y)\, \mathrm{e}^{b(y^2-x^2)}\, \mathrm{erfc}\left(\sqrt{\frac{2b}{1-\tau}}\,y\right)$$
$$\times \left(p_{j-1}(x+\mathrm{i}y)p_{k-1}(x-\mathrm{i}y) - p_{k-1}(x+\mathrm{i}y)p_{j-1}(x-\mathrm{i}y)\right). \tag{3.3}$$

For k, N *even we have*

$$Z_{k,(N-k)/2}[u,v] = \frac{(\sqrt{1+\tau})^{N(N-1)/2}}{2^{N(N+1)/4}\prod_{l=1}^{N}\Gamma(l/2)}[\zeta^{k/2}]\mathrm{Pf}[\zeta\alpha_{j,l}[u] + \beta_{j,l}[v]]_{j,l=1,\dots,N}, \tag{3.4}$$

where $[\zeta^p]f(\zeta)$ *denotes the coefficient of* ζ^p *in* $f(\zeta)$.

We remark that the Pfaffian operation applies to even-dimensional anti-symmetric matrices; formula (3.4) therefore requires modification for N odd. Such a modification is known [24], but to avoid having to consider separately the cases N even and N odd, only the case N even will be considered hereforth (it is planned to address the case N odd in a separate publication).

From the definitions, $Z_{k,(N-k)/2}[1,1] = p_{k,N}$ and so we have

$$p_{k,N} = \frac{(\sqrt{1+\tau})^{N(N-1)/2}}{2^{N(N+1)/4}\prod_{l=1}^{N}\Gamma(l/2)}[\zeta^{k/2}]\mathrm{Pf}[\zeta\alpha_{j,l}[1] + \beta_{j,l}[1]]_{j,l=1,\dots,N}\Big|_{b=1} \tag{3.5}$$

(it is valid to set $b = 1$ since as noted below (2.7) $p_{k,N}$ is independent of b). Suppose for definiteness that in (3.2) we choose $p_j(x) = x^j (j = 0, \dots, N-1)$. Changing variables $x \mapsto -x, y \mapsto -y$ shows

$$\alpha_{2j,2k}[1] = \alpha_{2j-1,2k-1}[1] = 0, \tag{3.6}$$

while we see by introducing polar coordinates and changing variables $\theta \mapsto \pi - \theta$ that furthermore

$$\beta_{2j,2k}[1] = \beta_{2j-1,2k-1}[1] = 0. \tag{3.7}$$

Equations (3.6) and (3.7) give that the matrix in (3.5) has a checkerboard pattern of zeros. Recalling the general formula $(\mathrm{Pf} A)^2 = \det A$, rearranging rows and columns in A so that all non-zero entries are in the top right and bottom right $N \times N$ blocks, and using the fact that the entries are anti-symmetric shows the Pfaffian can be written as a determinant of half its size. Thus

$$p_{k,N} = \frac{(\sqrt{1+\tau})^{N(N-1)/2}}{2^{N(N+1)/4}\prod_{l=1}^{N}\Gamma(l/2)}[\zeta^{k/2}]\det[\zeta\alpha_{2j-1,2k}[1] + \beta_{2j-1,2k}[1]]_{j,k=1,\dots,N/2}, \tag{3.8}$$

where the quantities in the determinant have $b = 1$.

Consider now the evaluation of the entries in determinant (3.8). In relation to $\alpha_{2j-1,2k}[1]$ integration by parts shows

$$\alpha_{2j-1,2k}[1] = 2(k-1)\alpha_{2j-1,2k-2}[1] + 2\Gamma(j+k-3/2)$$

and thus we obtain the explicit formula

$$\alpha_{2j-1,2k}[1] = 2^k(k-1)!\sum_{p=1}^{k}\frac{\Gamma(j+p-3/2)}{2^{p-1}(p-1)!}. \tag{3.9}$$

For $\beta_{2j-1,2k}[1]$ we see from the definition that

$$\begin{aligned}\beta_{2j-1,2k}[1] &= -4\,\mathrm{Im}\int_{\mathbb{R}^+}\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{y^2-x^2}\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau}}y\right)(x+\mathrm{i}y)^{2j-2}(x-\mathrm{i}y)^{2k-1}\\ &= -4\sum_{\substack{l=0\\ l+p\ \mathrm{odd}}}^{2j-2}\sum_{p=0}^{2k-1}\binom{2j-2}{l}\binom{2k-1}{p}(-1)^p\Gamma(j+k-1-(l+p)/2)I_{l+p},\end{aligned} \tag{3.10}$$

where

$$I_j := \int_0^\infty y^j\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau}}\right)\mathrm{e}^{y^2}\,\mathrm{d}y \qquad (j\ \text{odd}).$$

Integration by parts shows

$$I_j = -\frac{(j-1)}{2}I_{j-2} + \frac{1}{\sqrt{\pi}}\sqrt{\frac{2}{1-\tau}}\left(\frac{1-\tau}{1+\tau}\right)^{j/2}\frac{1}{2}\Gamma(j/2) - \frac{1}{2}\delta_{j,1}.$$

This recurrence has solution

$$I_j = \frac{(-1)^{(j-1)/2}((j-1)/2)!}{2}\left(\sqrt{\frac{2}{1+\tau}}\sum_{p=0}^{(j-1)/2}(-1)^p\left(\frac{1-\tau}{1+\tau}\right)^p\frac{(1/2)_p}{p!} - 1\right) \tag{3.11}$$

which makes $\beta_{2j-1,2k}[1]$ explicit. In the case $\tau = 0$ (3.11) reduces to a result of Edelman [6].

Some remarks on the calculation of $p_{k,N}$ in the case $\tau = 0$ obtained in earlier works are in order. The exact values of $p_{k,N}$, which are of the form $a + b\sqrt{2}$ with a and b being rational, for N up to 9 were first tabulated in [6], by direct integration of (2.5). The latter involves combining $N!$ terms, and so is not practical for larger N. A formula closely related to (3.5) was given in [18], thereby allowing for a tabulation in polynomial time in N, while formula (3.5) itself in the case $\tau = 0$ was given in [11]. A determinant formula equivalent to (3.8), but derived using a different strategy, is given in [1]. This latter work (see also [18]) gives that $p_{k,N}$ can be written as an elementary symmetric polynomial, expanded in the power sum basis, with entries read off from the entries of the corresponding case of Pfaffian (3.5). Such a result is a consequence of the so-called Pfaffian integration theorem [2] and also holds for general τ, although we do not pursue the details.

4. Correlations and skew orthogonal polynomials

As realizations of X are not conditioned on the number of real eigenvalues k, the generalized partition function $Z_N[u, v]$ appropriate for calculation of the correlation functions is obtained by summing (3.1) over all allowed k,

$$Z_N[u, v] = \sum_{\substack{k=0 \\ k \text{ even}}}^{N} Z_{k,(N-k)/2}[u, v].$$

It follows from (3.4) that

$$Z_N[u, v] = \frac{(\sqrt{1+\tau})^{N(N-1)/2}}{2^{N(N+1)/4} \prod_{l=1}^{N} \Gamma(l/2)} \operatorname{Pf}[\zeta \alpha_{j,l}[u] + \beta_{j,l}[v]]_{j,l=1,\ldots,N}. \tag{4.1}$$

Correlation functions can be calculated from this by functional differentiation. For example, the n-point correlation function between real eigenvalues at $x_1, \ldots, x_n$ is given by

$$\rho_{(n)}^{\mathrm{r}}(x_1, \ldots, x_n) = \frac{1}{Z_N[1, 1]} \frac{\delta^n}{\delta u(x_1) \cdots \delta u(x_n)} Z_N[u, 1]\bigg|_{u=1}. \tag{4.2}$$

Moreover, all n-point correlation functions can be expressed in terms of a $2k \times 2k$ Pfaffian [3] (see also [11] in the case of all real eigenvalues in the correlation, or all complex eigenvalues). The structure of these formulae is the same for all allowed τ and b, which is a consequence of the structure of the entries of (4.1) being the same for all allowed τ and b. In particular, in case (4.2)

$$\rho_{(n)}^{\mathrm{r}}(x_1, \ldots, x_n) = \operatorname{Pf}\begin{bmatrix} -\tilde{I}^{\mathrm{r}}(x_j, x_k) & S^{\mathrm{r}}(x_j, x_k) \\ -S^{\mathrm{r}}(x_k, x_j) & D^{\mathrm{r}}(x_j, x_k) \end{bmatrix}, \tag{4.3}$$

where with

$$\Phi_k(x) := \int_{-\infty}^{\infty} \operatorname{sgn}(x-y) p_k(y)\, \mathrm{e}^{-y^2/2(1+\tau)}\, \mathrm{d}y \tag{4.4}$$

one has

$$S^{\mathrm{r}}(x, y) = \sum_{k=0}^{N/2-1} \frac{\mathrm{e}^{-y^2/2(1+\tau)}}{u_k} (\Phi_{2k}(x) p_{2k+1}(y) - \Phi_{2k+1}(x) p_{2k}(y)) \tag{4.5}$$

$$D^{\mathrm{r}}(x, y) = \frac{\partial}{\partial x} S^{\mathrm{r}}(x, y), \qquad \tilde{I}^{\mathrm{r}}(x, y) = \frac{1}{2} \operatorname{sgn}(y-x) - \int_x^y S^{\mathrm{r}}(x, z)\, \mathrm{d}z. \tag{4.6}$$

We too note the explicit Pfaffian formula for the correlation between complex eigenvalues

$$\rho^{\mathrm{c}}_{(n)}((x_1,y_1),\ldots,(x_n,y_n)) = \prod_{l=1}^{n}\left(2\mathrm{i}\,\mathrm{e}^{(y_l^2-x_l^2)/(1+\tau)}\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau^2}}y_l\right)\right)$$
$$\times\,\mathrm{Pf}\begin{bmatrix} S^{\mathrm{c}}_\tau(\bar z_j,\bar z_k) & S^{\mathrm{c}}_\tau(\bar z_j,z_k)\\ S^{\mathrm{c}}_\tau(z_j,\bar z_k) & S^{\mathrm{c}}_\tau(z_j,z_k)\end{bmatrix}, \tag{4.7}$$

where $z_j := x_j + \mathrm{i}y_j$ and with $q_{2j-2}(z) := -p_{2j-1}(z)$, $q_{2j-1}(z) := p_{2j-2}(z)$,

$$S^{\mathrm{c}}_\tau(w,z) = \sum_{j=1}^{N}\frac{p_{j-1}(w)q_{j-1}(z)}{u_{[(j-1)/2]}}. \tag{4.8}$$

The dependence on τ comes in through the requirements of the polynomials $\{p_j(x)\}_{j=0,1,\ldots}$. In addition to being monic of the appropriate degree as in proposition 1, they must be skew orthogonal with respect to the skew inner product associated with the matrix in (4.1) for $u = v = 1$.

Explicitly, this inner product reads as

$$(f,g) := (f,g)_{\mathrm{r}} + (f,g)_{\mathrm{c}} \tag{4.9}$$

with

$$(f,g)_{\mathrm{r}} := \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-b(x^2+y^2)/2}f(x)g(y)\,\mathrm{sgn}(y-x)$$
$$(f,g)_{\mathrm{c}} := 2\mathrm{i}\int_{\mathbb{R}^2_+}\mathrm{d}x\,\mathrm{d}y\,\mathrm{e}^{b(y^2-x^2)}\,\mathrm{erfc}\left(\sqrt{\frac{2b}{1-\tau}}y\right)(f(x+\mathrm{i}y)g(x-\mathrm{i}y)-g(x+\mathrm{i}y)f(x-\mathrm{i}y)).$$

The set $\{p_j(x)\}_{j=0,1,\ldots}$ is said to be skew orthogonal if

$$(p_{2j},p_{2k}) = (p_{2j+1},p_{2k+1}) = 0$$
$$(j,k=0,1,\ldots)\qquad (p_{2j},p_{2k+1}) = 0$$
$$(j,k=0,1,\ldots\; j\neq k),$$

while $(p_{2j},p_{2j+1}) = u_j \neq 0$. The main technical task then is to compute these polynomials. Note that the parameter b acts as a scale of the coordinates, and so there is no loss of generality in setting b to a specific value. It turns out that a convenient choice is $b = 1/(1+\tau)$. Making this choice, the skew inner product of interest reads as

$$\langle f,g\rangle := \langle f,g\rangle_{\mathrm{r}} + \langle f,g\rangle_{\mathrm{c}} \tag{4.10}$$

with

$$\langle f,g\rangle_{\mathrm{r}} := \int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}}f(x)g(y)\,\mathrm{sgn}(y-x),$$
$$\langle f,g\rangle_{\mathrm{c}} := 2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x\int_{0}^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau^2}}y\right)$$
$$\times\,[f(x+\mathrm{i}y)g(x-\mathrm{i}y)-g(x+\mathrm{i}y)f(x-\mathrm{i}y)].$$

Theorem 1. *Introduce the scaled monic Hermite polynomials*

$$C_n(z) = \left(\frac{\tau}{2}\right)^{n/2}H_n\left(\frac{z}{\sqrt{2\tau}}\right). \tag{4.11}$$

The family of monic polynomials $\{R_j(z)\}_{j=0,1,\ldots}$ *with*

$$R_{2n+1}(z) = C_{2n+1}(z) - 2nC_{2n-1}(z), \qquad R_{2n}(z) = C_{2n}(z) \tag{4.12}$$

are skew orthogonal with respect to the skew inner product (4.10). Furthermore, the normalization r_n is given by

$$r_n := \langle R_{2n}, R_{2n+1} \rangle = (2n)!2\sqrt{2\pi}(1+\tau). \tag{4.13}$$

Before proceeding to the proof of this result, which will be done in the following section, we make note of two corollaries. The first is that the Hermite polynomial properties

$$\frac{\mathrm{d}}{\mathrm{d}z} C_n(z) = nC_{n-1}(z), \tag{4.14}$$

$$zC_n(z) = C_{n+1}(z) + n\tau C_{n-1}(z). \tag{4.15}$$

allow us to verify that the first of the two formulae in (4.12) can be rewritten to read as

$$R_{2n+1}(z) = -(1+\tau)\,\mathrm{e}^{z^2/2(1+\tau)} \frac{\mathrm{d}}{\mathrm{d}z}(\mathrm{e}^{-z^2/2(1+\tau)} C_{2n}(z)). \tag{4.16}$$

Use will be made of this form in the analysis of correlations (4.3). The second is that taking the limit $\tau \to 0$ in theorem 1 gives the family of skew orthogonal relevant to the real Ginibre ensemble, a result which was announced and made use of in [11].

Corollary 1. *The family of monic polynomials $\{p_j(z)\}_{j=0,1,\ldots}$ specified by*

$$p_{2n+1}(z) = z^{2n+1} - 2nz^{2n-1}, \qquad p_{2n}(z) = z^{2n} \tag{4.17}$$

are skew orthogonal with respect to the skew inner product (4.10) in the case $\tau = 0$. The corresponding normalization is given by

$$u_n := \langle p_{2n}, p_{2n+1} \rangle = (2n)!2\sqrt{2\pi}. \tag{4.18}$$

5. Proof of theorem 1

Note that $R_{2n+1}(z)$ is an odd polynomial while $R_{2n}(z)$ is even. These properties are sufficient for the derivation of (3.6) and (3.7) so it is immediate that

$$\langle R_{2j}, R_{2k} \rangle = \langle R_{2j+1}, R_{2k+1} \rangle = 0. \tag{5.1}$$

It remains to verify that

$$\langle R_{2j+1}, R_{2k} \rangle = 0 \tag{5.2}$$

for $k \neq j$, and that for $k = j$ normalization (4.13) results. This will be done by computing the explicit form of the skew inner product between C_{2j+1} and C_{2k},

$$\langle C_{2j+1}, C_{2k} \rangle = \begin{cases} -2^{j+k+(3/2)} j!\Gamma\left(k+\frac{1}{2}\right)(1+\tau), & j \geqslant k, \\ 0, & j < k. \end{cases} \tag{5.3}$$

Assuming (5.3), the explicit formulae (4.12) show that (5.2) is valid and furthermore give normalization (4.13). The task is thus reduced to proving (5.3). For this, repeated use will be made of properties (4.14), (4.15) of the Hermite polynomials (4.11).

Let us first define

$$I_{j,k} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) C_{2k}(y)\,\mathrm{sgn}(y-x). \tag{5.4}$$

Then a partial integration over x gives

$$I_{j,k} = \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{y^2}{2(1+\tau)}} C_{2k}(y) \left[\mathrm{e}^{-\frac{x^2}{2(1+\tau)}} \frac{C_{2j+2}(x)}{2j+2} \operatorname{sgn}(y-x) \right]_{x=-\infty}^{x=\infty}$$
$$+ \frac{1}{1+\tau} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} \frac{xC_{2j+2}(x)}{2j+2} C_{2k}(y) \operatorname{sgn}(y-x)$$
$$+ \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} \frac{C_{2j+2}(x)}{2j+2} C_{2k}(y) 2\delta(y-x)$$
$$= \frac{1}{1+\tau} \frac{1}{2j+2} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+3}(x) C_{2k}(y) \operatorname{sgn}(y-x)$$
$$+ \frac{\tau}{1+\tau} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) C_{2k}(y) \operatorname{sgn}(y-x)$$
$$+ \frac{2}{2j+2} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+2}(x) C_{2k}(x)$$
$$= \frac{1}{1+\tau} \frac{1}{2j+2} I_{j+1,k} + \frac{\tau}{1+\tau} I_{j,k} + \frac{2}{2j+2} \frac{\xi_{j,k}}{1+\tau}, \tag{5.5}$$

where

$$\xi_{j,k} = (1+\tau) \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+2}(x) C_{2k}(x). \tag{5.6}$$

Similarly a partial integration over y gives

$$I_{j,k} = \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{2(1+\tau)}} C_{2j+1}(x) \left[\mathrm{e}^{-\frac{y^2}{2(1+\tau)}} \frac{C_{2k+1}(y)}{2k+1} \operatorname{sgn}(y-x) \right]_{y=-\infty}^{y=\infty}$$
$$+ \frac{1}{1+\tau} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) \frac{yC_{2k+1}(y)}{2k+1} \operatorname{sgn}(y-x)$$
$$- \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) \frac{C_{2k+1}(y)}{2k+1} 2\delta(y-x)$$
$$= \frac{1}{1+\tau} \frac{1}{2k+1} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) C_{2k+2}(y) \operatorname{sgn}(y-x)$$
$$\frac{\tau}{1+\tau} \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2+y^2}{2(1+\tau)}} C_{2j+1}(x) C_{2k}(y) \operatorname{sgn}(y-x)$$
$$- \frac{2}{2k+1} \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+1}(x) C_{2k+1}(x)$$
$$= \frac{1}{1+\tau} \frac{1}{2k+1} I_{j,k+1} + \frac{\tau}{1+\tau} I_{j,k} - \frac{2}{2k+1} \frac{\eta_{j,k}}{1+\tau}, \tag{5.7}$$

where

$$\eta_{j,k} = (1+\tau) \int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+1}(x) C_{2k+1}(x). \tag{5.8}$$

Thus we obtain recursion relations

$$I_{j+1,k} = (2j+2) I_{j,k} - 2\xi_{j,k}, \qquad I_{j,k+1} = (2k+1) I_{j,k} + 2\eta_{j,k}. \tag{5.9}$$

Let us next derive the recursion relations for

$$J_{j,k} = 2\mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x \int_{0}^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}} \operatorname{erfc}(\gamma y)$$
$$\times [C_{2j+1}(x+\mathrm{i}y) C_{2k}(x-\mathrm{i}y) - C_{2k}(x+\mathrm{i}y) C_{2j+1}(x-\mathrm{i}y)] \tag{5.10}$$

with

$$\gamma = \sqrt{\frac{2}{1-\tau^2}}. \tag{5.11}$$

For that purpose, we consider an integral

$$K_{j,k} = 2\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) \\ \times [C_{2j+2}(x+\mathrm{i}y)C_{2k-1}(x-\mathrm{i}y) - C_{2k-1}(x+\mathrm{i}y)C_{2j+2}(x-\mathrm{i}y)]. \tag{5.12}$$

For $k \geqslant 1$, a partial integration over x gives

$$\begin{aligned} K_{j,k} = 2\mathrm{i}\int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) & \left[\mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+2}(x+\mathrm{i}y)\frac{C_{2k}(x-\mathrm{i}y)}{2k}\right]_{x=-\infty}^{x=\infty} \\ & + 4\mathrm{i}\frac{1}{1+\tau}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) x C_{2j+2}(x+\mathrm{i}y)\frac{C_{2k}(x-\mathrm{i}y)}{2k} \\ & - 2\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y)(2j+2)C_{2j+1}(x+\mathrm{i}y)\frac{C_{2k}(x-\mathrm{i}y)}{2k} + c.c. \\ & = \frac{2\mathrm{i}}{k}\frac{1}{1+\tau}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) x C_{2j+2}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) \\ & - 2\mathrm{i}\frac{j+1}{k}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) + c.c. \end{aligned} \tag{5.13}$$

On the other hand, a partial integration over y gives

$$\begin{aligned} K_{j,k} = 2\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} & \left[\mathrm{e}^{\frac{y^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) C_{2j+2}(x+\mathrm{i}y)\frac{\mathrm{i}C_{2k}(x-\mathrm{i}y)}{2k}\right]_{y=0}^{y=\infty} \\ & - 4\mathrm{i}\frac{1}{1+\tau}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) y C_{2j+2}(x+\mathrm{i}y)\frac{\mathrm{i}C_{2k}(x-\mathrm{i}y)}{2k} \\ & + 2\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y)(2j+2)C_{2j+1}(x+\mathrm{i}y)\frac{C_{2k}(x-\mathrm{i}y)}{2k} \\ & + 2\mathrm{i}\frac{2\gamma}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{e}^{-(\gamma y)^2} C_{2j+2}(x+\mathrm{i}y)\frac{\mathrm{i}C_{2k}(x-\mathrm{i}y)}{2k} + c.c. \\ & + \frac{1}{k}\int_{-\infty}^{\infty} \mathrm{d}x\, \mathrm{e}^{-\frac{x^2}{1+\tau}} C_{2j+2}(x)C_{2k}(x) \\ & - \frac{2\mathrm{i}}{k}\frac{1}{1+\tau}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y)\, \mathrm{i}y C_{2j+2}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) \\ & + 2\mathrm{i}\frac{j+1}{k}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\, \mathrm{erfc}(\gamma y) C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) \\ & - \frac{2}{k}\sqrt{\frac{2}{\pi(1-\tau^2)}}\int_{-\infty}^{\infty} \mathrm{d}x \int_0^{\infty} \mathrm{d}y\, \mathrm{e}^{-\frac{x^2}{1+\tau}-\frac{y^2}{1-\tau}} C_{2j+2}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) + c.c. \end{aligned} \tag{5.14}$$

Comparing (5.13) and (5.14), we obtain

$$\begin{aligned}
&\frac{2\mathrm{i}}{k}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)(x+\mathrm{i}y)C_{2j+2}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y)\\
&\qquad -4\mathrm{i}\frac{j+1}{k}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y)^{2k}+c.c.\\
&\quad=\frac{2}{k}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\frac{x^2}{1+\tau}}C_{2j+2}(x)C_{2k}(x)\\
&\qquad -\frac{2}{k}\sqrt{\frac{2}{\pi(1-\tau^2)}}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-\frac{x^2}{1+\tau}-\frac{y^2}{1-\tau}}\\
&\qquad\times[C_{2j+2}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y)+C_{2j+2}(x-\mathrm{i}y)C_{2k}(x+\mathrm{i}y)].
\end{aligned}\tag{5.15}$$

Therefore, noting (4.15) and the orthogonality relation (see e.g. [12])

$$\int_{-\infty}^{\infty}\mathrm{d}x\int_{-\infty}^{\infty}\mathrm{d}y\,\mathrm{e}^{-\frac{x^2}{1+\tau}}\,\mathrm{e}^{-\frac{y^2}{1-\tau}}C_m(x+\mathrm{i}y)C_n(x-\mathrm{i}y)=\pi m!\sqrt{1-\tau^2}\delta_{m,n},\tag{5.16}$$

we can derive

$$J_{j+1,k}=(2j+2)J_{j,k}+2\xi_{j,k}-2\sqrt{2\pi}(2j+2)!(1+\tau)\delta_{j+1,k}.\tag{5.17}$$

In order to derive another recursion relation, we similarly employ partial integrations to find

$$\begin{aligned}
K_{j-1,k+1}&=2\mathrm{i}\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\left[\mathrm{e}^{-\frac{x^2}{1+\tau}}\frac{C_{2j+1}(x+\mathrm{i}y)}{2j+1}C_{2k+1}(x-\mathrm{i}y)\right]_{x=-\infty}^{x=\infty}\\
&\quad+4\mathrm{i}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\frac{C_{2j+1}(x+\mathrm{i}y)}{2j+1}xC_{2k+1}(x-\mathrm{i}y)\\
&\quad-2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\frac{C_{2j+1}(x+\mathrm{i}y)}{2j+1}(2k+1)C_{2k}(x-\mathrm{i}y)+c.c.\\
&=\frac{4\mathrm{i}}{2j+1}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)xC_{2k+1}(x-\mathrm{i}y)\\
&\quad-2\mathrm{i}\frac{2k+1}{2j+1}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y)+c.c.
\end{aligned}\tag{5.18}$$

and

$$\begin{aligned}
K_{j-1,k+1}&=2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\frac{x^2}{1+\tau}}\left[\mathrm{e}^{\frac{y^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\frac{C_{2j+1}(x+\mathrm{i}y)}{i(2j+1)}C_{2k+1}(x-\mathrm{i}y)\right]_{y=0}^{y=\infty}\\
&\quad-4\mathrm{i}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\frac{C_{2j+1}(x+\mathrm{i}y)}{\mathrm{i}(2j+1)}yC_{2k+1}(x-\mathrm{i}y)\\
&\quad+2\mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)\frac{C_{2j+1}(x+\mathrm{i}y)}{2j+1}(2k+1)C_{2k}(x-\mathrm{i}y)\\
&\quad+2\mathrm{i}\frac{2\gamma}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{e}^{-(\gamma y)^2}\frac{C_{2j+1}(x+\mathrm{i}y)}{\mathrm{i}(2j+1)}C_{2k+1}(x-\mathrm{i}y)+c.c.\\
&=-\frac{2}{2j+1}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\frac{x^2}{1+\tau}}C_{2j+1}(x)C_{2k+1}(x)\\
&\quad+\frac{4\mathrm{i}}{2j+1}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)\,\mathrm{i}yC_{2k+1}(x-\mathrm{i}y)
\end{aligned}$$

$$+ 2\mathrm{i}\frac{2k+1}{2j+1}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y)$$
$$+ \frac{4}{2j+1}\sqrt{\frac{2}{\pi(1-\tau^2)}}\int_{-\infty}^{\infty}\mathrm{d}x$$
$$\times\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-\frac{x^2}{1+\tau}-\frac{y^2}{1-\tau}}C_{2j+1}(x+\mathrm{i}y)C_{2k+1}(x-\mathrm{i}y) + c.c. \tag{5.19}$$

A comparison of (5.18) and (5.19) yields

$$\frac{4\mathrm{i}}{2j+1}\frac{1}{1+\tau}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)(x-\mathrm{i}y)C_{2k+1}(x-\mathrm{i}y)$$
$$- 4\mathrm{i}\frac{2k+1}{2j+1}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{\frac{y^2-x^2}{1+\tau}}\,\mathrm{erfc}(\gamma y)C_{2j+1}(x+\mathrm{i}y)C_{2k}(x-\mathrm{i}y) + c.c.$$
$$= -\frac{4}{2j+1}\int_{-\infty}^{\infty}\mathrm{d}x\,\mathrm{e}^{-\frac{x^2}{1+\tau}}C_{2j+1}(x)C_{2k+1}(x)$$
$$+ \frac{4}{2j+1}\sqrt{\frac{2}{\pi(1-\tau^2)}}\int_{-\infty}^{\infty}\mathrm{d}x\int_0^{\infty}\mathrm{d}y\,\mathrm{e}^{-\frac{x^2}{1+\tau}-\frac{y^2}{1-\tau}}$$
$$\times[C_{2j+1}(x+\mathrm{i}y)C_{2k+1}(x-\mathrm{i}y) + C_{2j+1}(x-\mathrm{i}y)C_{2k+1}(x+\mathrm{i}y)]. \tag{5.20}$$

As before it follows that

$$J_{j,k+1} = (2k+1)J_{j,k} - 2\eta_{j,k} + 2\sqrt{2\pi}(2j+1)!(1+\tau)\delta_{j,k}. \tag{5.21}$$

Let us employ the notation

$$L_{j,k} = I_{j,k} + J_{j,k}. \tag{5.22}$$

Then, from (5.9), (5.17) and (5.21), we obtain the recursion relations

$$\begin{aligned} L_{j+1,k} &= (2j+2)L_{j,k} - 2\sqrt{2\pi}(2j+2)!(1+\tau)\delta_{j+1,k}, \qquad j\geqslant 0, k\geqslant 1,\\ L_{j,k+1} &= (2k+1)L_{j,k} + 2\sqrt{2\pi}(2j+1)!(1+\tau)\delta_{j,k}, \qquad j\geqslant 0, k\geqslant 0. \end{aligned} \tag{5.23}$$

Using these recursion relations and noting

$$L_{0,0} = -2\sqrt{2\pi}(1+\tau), \tag{5.24}$$

we can readily find that (5.3) holds.

6. Asymptotic properties of the correlations

6.1. Eigenvalue support

The eigenvalue support for the real Ginibre ensemble ($\tau = 0$ case) is to leading order a circle of radius $\sqrt{N}$. To gain some insight into its expected form for $0 \leqslant \tau < 1$, consider the portion of (2.7) which is dependent on $\{z_j := x_j + \mathrm{i}y_j\}_{j=1,\ldots,(N-k)/2}$. For y_j large this portion is proportional to (1.2) with $N \mapsto (N-k)/2$. As remarked below the latter equation, previous analysis of the one-point correlation has revealed that for PDF (1.2) the density is supported on an ellipse with semi-axes A and B given by (1.3). The exact results obtained above can be combined with the analysis of [12] to verify that this result persists in the present setting.

In [12] the boundary of the support is characterized by the values of (x, y) which maximize the difference

$$\rho^{\mathrm{c}}_{(1)}((x,y))|_{N\mapsto N+1} - \rho^{\mathrm{c}}_{(1)}((x,y)) \tag{6.1}$$

for large N. We know from (4.7) and (4.8) that

$$\rho^{c}_{(1)}((x,y)) = 2\mathrm{i}\,\mathrm{e}^{(y^2-x^2)/(1+\tau)}\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau^2}}\,y\right) S^{c}_{\tau}(\bar{z},z). \tag{6.2}$$

To compute the asymptotic form of (6.1), following the beginnings of a strategy used to analyze the two-point correlation for (1.2) in [10], use will be made of an integral form of (6.2). In this regard, from a standard integral representation of the Hermite polynomials we have

$$C_n(z) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \mathrm{e}^{-t^2}(z+\sqrt{2\tau}\,\mathrm{i}t)^n\,\mathrm{d}t. \tag{6.3}$$

It follows from this and corollary 1 that

$$S^{c}_{\tau}(w,z) = \frac{1}{\pi(1+\tau)}\int_{-\infty}^{\infty}\mathrm{d}t_1\,\mathrm{e}^{-t_1^2}\int_{-\infty}^{\infty}\mathrm{d}t_2\,\mathrm{e}^{-t_2^2}\,S^{c}_{0}(w+\sqrt{2\tau}\,\mathrm{i}t_1, z+\sqrt{2\tau}\,\mathrm{i}t_2). \tag{6.4}$$

On the other hand, substituting (4.17) into (4.8) gives for S^{c}_{0} the simple expression

$$S^{c}_{0}(w,z) = \frac{w-z}{2\sqrt{2\pi}}\sum_{j=0}^{N-2}\frac{(wz)^j}{\Gamma(j+1)}. \tag{6.5}$$

Substituting (6.5) into (6.4) and making further use of (6.3) it follows that for large N, x, y

$$\rho^{c}_{(1)}((x,y))|_{N\mapsto N+1} - \rho^{c}_{(1)}((x,y)) \sim \frac{\sqrt{2}}{\pi(1+\tau)}\frac{1}{(N-2)!}\exp\left(-\frac{1}{1-\tau^2}\left((x^2+y^2)-\tau(x^2-y^2)\right)\right)|C_{N-2}(z)|^2.$$

This same function of x, y and N results from studying difference (6.1) in the case of PDF (1.2). As remarked above, working in [12] deduces from this that the boundary of the support is given by an ellipse with semi-axes specified by (1.3).

6.2. *Density of real eigenvalues*

Next asymptotic properties of the density of real eigenvalues will be considered. According to (4.3) and (4.5)

$$\rho^{r}_{(1)}(x) = \frac{\mathrm{e}^{-x^2/2(1+\tau)}}{2\sqrt{2\pi}(1+\tau)}\sum_{k=0}^{N/2-1}\frac{1}{(2k)!}\left(\Phi_{2k}(x)R_{2k+1}(x) - \Phi_{2k+1}(x)R_{2k}(x)\right). \tag{6.6}$$

The mean number of real eigenvalues is obtained by integrating $\rho^{r}_{(1)}(x)$ over the real line. Making use of (6.6), (4.4) (with p_k replaced by R_k), (4.16) and theorem 1 shows

$$\int_{-\infty}^{\infty}\rho^{r}_{(1)}(x)\,\mathrm{d}x = 2\sqrt{\frac{\tau}{\pi}}\sum_{k=0}^{N/2-1}\frac{1}{(2k)!}\left(\frac{\tau}{2}\right)^{2k}\int_{-\infty}^{\infty}\mathrm{e}^{-2\tau x^2/(1+\tau)}\,(H_{2k}(x))^2\,\mathrm{d}x. \tag{6.7}$$

Further, use of a tabulated integral [15, section 7.373] and a Kummer transformation for ${}_2F_1$, gives

$$\int_{-\infty}^{\infty}\mathrm{e}^{-2\tau x^2/(1+\tau)}(H_{2k}(x))^2\,\mathrm{d}x = 2^{2k-1/2}\left(\frac{1+\tau}{1-\tau}\right)^{1/2}\tau^{-2k-1/2}\Gamma(2k+1/2) \times {}_2F_1(1/2,1/2;-2k+1/2;-\tau/(1-\tau)). \tag{6.8}$$

For large k the ${}_2F_1$ function is to leading order equal to unity. Hence the leading-order behavior of a general term in (6.7) is

$$\sqrt{\frac{1}{2\pi}}\left(\frac{1+\tau}{1-\tau}\right)^{1/2}\frac{1}{(2k)^{1/2}}$$

and so for large N

$$\int_{-\infty}^{\infty}\rho_{(1)}^{\rm r}(x)\,{\rm d}x \sim \sqrt{\frac{2N}{\pi}}\left(\frac{1+\tau}{1-\tau}\right)^{1/2}. \tag{6.9}$$

Substituting (6.8) into (6.7) and taking $\tau\to 0$ shows

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty}\rho_{(1)}^{(r)}(x)\,{\rm d}x = \sqrt{\frac{2}{\pi}}\sum_{k=0}^{N/2-1}\frac{\Gamma(2k+1/2)}{(2k)!} = \sqrt{2}\sum_{k=0}^{N/2-1}\frac{(4k-1)!!}{(4k)!!}, \tag{6.10}$$

where the second equality follows upon use of the duplication formula for $\Gamma(z)$. Result (6.10) was first obtained in [7].

To analyze the density itself for large N, we again make use of (4.4) (with p_k replaced by R_k), as well as the identity

$$R_{2k+2}(x)-(2k+1)R_{2k}(x) = -(1+\tau)\,{\rm e}^{x^2/2(1+\tau)}\frac{\rm d}{{\rm d}x}\left({\rm e}^{-x^2/2(1+\tau)}C_{2k+1}(x)\right)$$

(cf (4.16); this can be verified using (4.14), (4.15)) to rewrite (6.6) in the simplified form

$$\rho_{(1)}^{\rm r}(x) = \frac{{\rm e}^{-x^2/(1+\tau)}}{\sqrt{2\pi}}\sum_{k=0}^{N-2}\frac{1}{k!}(C_k(x))^2 + \frac{{\rm e}^{-x^2/2(1+\tau)}}{2\sqrt{2\pi}(1+\tau)}\frac{C_{N-1}(x)\Phi_{N-2}(x)}{(N-2)!}. \tag{6.11}$$

Use can now be made of the classical summation formula

$$\sum_{k=0}^{\infty}\frac{t^kH_k(x)H_k(y)}{k!2^k} = (1-t^2)^{-1/2}\,{\rm e}^{-t^2(x^2+y^2)/(1-t^2)}\,{\rm e}^{2xyt/(1-t^2)}, \qquad |t|<1 \tag{6.12}$$

to conclude

$$\rho_{(1)}^{\rm bulk}(x) := \lim_{N\to\infty}\rho_{(1)}^{\rm r}(x) = \frac{1}{\sqrt{2\pi(1-\tau^2)}}. \tag{6.13}$$

With the leading-order support of the real eigenvalues the interval $[-\sqrt{N}(1+\tau),\sqrt{N}(1+\tau)]$, to leading order the mean number of eigenvalues must be equal to $2\sqrt{N}(1+\tau)$ (the length of this interval) times density (6.13). This reclaims (6.9).

We turn our attention now to the neighborhood of the spectrum edge. In the case of the real Ginibre ensemble ($\tau=0$) the explicit form of the density profile about the spectrum edge at $x=\sqrt{N}$ was exhibited as [11]

$$\lim_{N\to\infty}\rho_{(1)}^{\rm r}(\sqrt{N}+X)\Big|_{\tau=0} = \frac{1}{\sqrt{2\pi}}\left(\frac{1}{2}(1-{\rm erf}\sqrt{2}X)+\frac{{\rm e}^{-X^2}}{2\sqrt{2}}(1+{\rm erf}X)\right). \tag{6.14}$$

For general $0\leqslant\tau<1$ the density at the spectrum edge is analyzed by setting $x=(1+\tau)\sqrt{N}+X$ in (6.11) then taking the limit $N\to\infty$. As is distinct from the bulk scaling, the summation and the term distinct from the summation both give $O(1)$ contributions. Consider first the latter.

To calculate the explicit form of the contributions, our main tool is the Plancherel–Rotach asymptotic formula [5, 22]

$$H_n(x)\sim(2n)^n\exp\left(\frac{x^2-x\sqrt{x^2-2n}-n}{2}-n\log(x-\sqrt{x^2-2n})\right)$$
$$\times\sqrt{\frac{1}{2}\left(1+\frac{x}{\sqrt{x^2-2n}}\right)} \tag{6.15}$$

valid for n large and $x > \sqrt{2n}$. Recalling (4.11) we require this formula with $x = \sqrt{N/2}(\sqrt{\tau} + 1/\sqrt{\tau}) + X/\sqrt{2\tau}, n = N - k$. A straightforward but tedious calculation gives that for k fixed

$$H_{N-k}(x) \sim (2N)^{N-k}\,\mathrm{e}^{-k}(1-\tau)^{-1/2} \times \exp\left(\frac{N}{2}(\tau - \log 2N - \log\tau) + \sqrt{N}X - \frac{X^2}{2(1-\tau)} + k + \frac{k}{2}\log 2N + \frac{k}{2}\log\tau\right). \tag{6.16}$$

Consider now the final term in (6.11). The asymptotic form of $C_{N-1}(x)$ can be read off (6.15) by setting $k = 1$. To use it to deduce the asymptotic form of $\Phi_{N-2}(x)$ we require the integral evaluation [15]

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2} H_{2m}(xy)\,\mathrm{d}x = \sqrt{\pi}\frac{(2m)!}{m!}(y^2-1)^m.$$

This formula allows us to write

$$\Phi_{N-2}(x) = \left(\frac{\tau}{2}\right)^{N/2-1}\left(\sqrt{2\pi(1+\tau)}\frac{(N-2)!}{(N/2-1)!}\tau^{1-N/2} - 2\int_x^{\infty}\mathrm{e}^{-t^2/2(1+\tau)}H_{N-2}\left(\frac{t}{\sqrt{2\tau}}\right)\mathrm{d}t\right). \tag{6.17}$$

It is in this form that we substitute (6.16) with $k = 2$. Combining results and making use too of Stirling's formula allows us to compute the sought limiting form,

$$\lim_{N\to\infty}\left(\frac{\mathrm{e}^{-x^2/2(1+\tau)}}{2\sqrt{2\pi}(1+\tau)}\frac{C_{N-1}(x)\Phi_{N-2}(x)}{(N-2)!}\bigg|_{x=(1+\tau)\sqrt{N}+X}\right) = \frac{1}{(1-\tau^2)^{1/2}}\frac{\mathrm{e}^{-X^2/(1-\tau^2)}}{4\sqrt{\pi}}(1+\mathrm{erf}(X/\sqrt{1-\tau^2})). \tag{6.18}$$

To analyze the sum in (6.11) the asymptotic expansion (6.16) must be extended to include terms $O(k/\sqrt{N})$. One finds these terms to be the multiplicative factor

$$\exp\left(-\frac{k^2\tau}{2N(1-\tau)} - \frac{kX}{\sqrt{N}(1-\tau)}\right).$$

Noting too from Stirling's formula that

$$\frac{1}{(N-k)!} \sim (2\pi N)^{1/2}\exp\left(N\log N - N - k\log N + \frac{k^2}{2N}\right),$$

after rearranging the order of summation so that $k \mapsto N - k (k = 2, \ldots, N)$ and recognizing that a Riemann sum approximation to definite integral results we find

$$\lim_{N\to\infty}\frac{\mathrm{e}^{-x^2/(1+\tau)}}{\sqrt{2\pi}}\sum_{k=0}^{N-2}\frac{1}{k!}(C_k(x))^2\bigg|_{x=(1+\tau)\sqrt{N}+X} = \frac{1}{(1-\tau^2)^{1/2}}\frac{1}{2\sqrt{2\pi}}(1-\mathrm{erf}(\sqrt{2}X/(1-\tau^2)^{1/2})). \tag{6.19}$$

Now adding together (6.18) and (6.19) gives for the edge density

$$\rho_{(1)}^{\mathrm{edge}}(X) := \lim_{N\to\infty}\rho_{(1)}^{\mathrm{r}}((1+\tau)\sqrt{N}+X) = \frac{1}{\sqrt{2\pi(1-\tau^2)}} \times\left(\frac{1}{2}(1-\mathrm{erf}(\sqrt{2}X/(1-\tau^2)^{1/2})) + \frac{\mathrm{e}^{-X^2/(1-\tau^2)}}{2\sqrt{2}}(1+\mathrm{erf}(X/(1-\tau^2)^{1/2}))\right). \tag{6.20}$$

Note that this agrees with (6.14) in the case $\tau = 0$.

We observe from (6.20) that $\rho_{(1)}^{\mathrm{edge}}(X)\,\mathrm{d}X$ for general $0 \leqslant \tau < 1$ is obtained from the case $\tau = 0$ by the simple scaling $X \mapsto X/\sqrt{1-\tau^2}$. Note that this same rule is valid for the bulk density of real eigenvalues (6.13). Indeed it is reasonable to expect that all local correlations are only altered by this change of scale, as the point process for general $0 \leqslant \tau < 1$ is locally identical to that for the point process in the case $\tau = 0$, except that the two-dimensional bulk density is scaled by a factor of $1/(1-\tau^2)$. We will now proceed to exhibit this fact for the general real–real and complex–complex correlations in the bulk.

6.3. k-point correlations in the bulk

Consider first the complex–complex case. Setting

$$\hat{S}_\tau^{\mathrm{c}}(w,z) := \mathrm{e}^{-(z^2+w^2)/2(1+\tau)} S_\tau^{\mathrm{c}}(w,z),$$

we see that (4.7) can be rewritten

$$\rho_{(n)}^{\mathrm{c}}((x_1,y_1),\ldots,(x_n,y_n)) = \prod_{l=1}^n \left(2\mathrm{i}\,\mathrm{erfc}\left(\sqrt{\frac{2}{1-\tau^2}}y_l\right)\right) \mathrm{Pf}\begin{bmatrix} \hat{S}_\tau^{\mathrm{c}}(\bar{z}_j,\bar{z}_k) & \hat{S}_\tau^{\mathrm{c}}(\bar{z}_j,z_k) \\ \hat{S}_\tau^{\mathrm{c}}(z_j,\bar{z}_k) & \hat{S}_\tau^{\mathrm{c}}(z_j,z_k) \end{bmatrix}. \tag{6.21}$$

Now, it is immediate from (6.5) that

$$\lim_{N\to\infty} S_0^{\mathrm{c}}(w,z) = \frac{w-z}{2\sqrt{2\pi}}\,\mathrm{e}^{wz}.$$

Substituting this into (6.3) and computing the resulting Gaussian integrals gives

$$\lim_{N\to\infty} S_\tau^{\mathrm{c}}(w,z) = \frac{(w-z)}{2\sqrt{2\pi}(1-\tau^2)} \exp\left(-\frac{\tau}{2(1-\tau^2)}(z^2+w^2) + \frac{1}{1-\tau^2}zw\right). \tag{6.22}$$

Consequently

$$\lim_{N\to\infty} \hat{S}_\tau^{\mathrm{c}}(w,z) = \frac{(w-z)}{2\sqrt{2\pi}(1-\tau^2)} \exp\left(-\frac{(z-w)^2}{2(1-\tau^2)}\right). \tag{6.23}$$

Substituting into (6.21) gives the bulk complex–complex correlations. The feature that the bulk limiting value of

$$\rho_{(n)}^{\mathrm{c}}((x_1,y_1),\ldots,(x_n,y_n)) \prod_{l=1}^n \mathrm{d}x_l\,\mathrm{d}y_l \tag{6.24}$$

for general $0 \leqslant \tau < 1$ is gotten from the $\tau = 0$ case by the replacements

$$(x_l,y_l) \mapsto (x_l/\sqrt{1-\tau^2},\, y_l/\sqrt{1-\tau^2})$$

is evident.

It remains to consider the real–real case. Proceeding as in the derivation of (6.11) shows that (4.5) can be rewritten

$$S^{\mathrm{r}}(x,y) = \frac{\mathrm{e}^{-(x^2+y^2)/2(1+\tau)}}{\sqrt{2\pi}} \sum_{k=0}^{N-2} \frac{1}{k!} C_k(x)C_k(y) + \frac{\mathrm{e}^{-y^2/2(1+\tau)}}{2\sqrt{2\pi}(1+\tau)} \frac{C_{N-1}(y)\Phi_{N-2}(x)}{(N-2)!} \tag{6.25}$$

As for the derivation of (6.13), we use (6.12) to both deduce that the final term vanishes as $N \to \infty$ (a consequence of the convergence of the sum) and to give a closed form evaluation of the summation. It follows that

$$\lim_{N\to\infty} S^{\mathrm{r}}(x,y) = \frac{1}{\sqrt{2\pi(1-\tau^2)}}\,\mathrm{e}^{-(x-y)^2/2(1-\tau^2)}. \tag{6.26}$$

In view of formulae (4.6) for the remaining quantities in (4.3), as for (6.24) we have that the limiting bulk value of

$$\rho^{\rm r}_{(n)}(x_1, \ldots, x_n)\, {\rm d}x_1 \cdots {\rm d}x_n$$

for general $0 \leqslant \tau < 1$ is gotten from the $\tau = 0$ case by the replacements $x_j \mapsto x_j/\sqrt{1-\tau^2}$.

6.4. The weakly non-symmetric limit

In relation to the ensemble interpolating between the complex Ginibre ensemble and the GUE, it was exhibited in [13] that well-defined correlations result by setting $\tau = 1 - \alpha^2/N$, then taking $N \to \infty$. Similarly, scaled correlations were computed in this limit for the ensemble interpolating between the real quaternion Ginibre ensemble and the GSE [17]. Here the scaled correlations of the real Ginibre/ GOE interpolating ensemble will be calculated. Note that with this scaling (1.3) gives that the eigenvalue support collapses onto the interval $[-2\sqrt{N}, 2\sqrt{N}]$ of the real axis. The mean spacing between eigenvalues is then $O(1/\sqrt{N})$, suggesting that we should also multiply coordinates by $\pi/\sqrt{N}$ (the proportionality π is chosen for convenience, then a unit real density results) before taking $N \to \infty$.

Now, using the asymptotic expansion

$$\frac{\Gamma(n/2+1)}{\Gamma(n+1)}\, {\rm e}^{-x^2} H_n(x) = \cos(\sqrt{2n+1}x - n\pi/2) + O(n^{-1/2}),$$

we deduce from (6.25) that

$$\frac{\pi}{\sqrt{N}} S^{\rm r}_\tau\left(\frac{\pi x}{\sqrt{N}}, \frac{\pi y}{\sqrt{N}}\right)\bigg|_{\tau=1-\alpha^2/2N} \sim \frac{1}{2}\sqrt{\frac{\pi}{2N}} \sum_{k=0}^{N-2} \frac{{\rm e}^{-k\alpha^2/N} k!}{2^k((k/2)!)^2} \cos\left(\pi\sqrt{\frac{k}{N}}(x-y)\right).$$

Making use of Stirling's formula, a Riemann sum approximation to a definite integral is obtained, and we compute

$$\lim_{N\to\infty} \frac{\pi}{\sqrt{N}} S^{\rm r}_\tau\left(\frac{\pi x}{\sqrt{N}}, \frac{\pi y}{\sqrt{N}}\right)\bigg|_{\tau=1-\alpha^2/N} = \int_0^1 {\rm e}^{-\alpha^2 u^2} \cos \pi u(x-y)\, {\rm d}u. \tag{6.27}$$

Note that in contrast to the correlations implied by (6.26), we see from (6.27) that the correlations in the present setting of the weakly non-symmetric limit exhibit an algebraic decay. For the limiting form of the complex–complex correlations, we first note from (6.3)–(6.5) that

$$S^{\rm c}_\tau(w,z) = \frac{1}{2(1+\tau)\sqrt{2\pi}} \sum_{j=0}^{N-2} \frac{C_{j+1}(w)C_j(z) - C_j(w)C_{j+1}(z)}{\Gamma(j+1)}.$$

Proceeding now as for the working which lead to (6.27) shows

$$\lim_{N\to\infty} \left(\frac{\pi}{\sqrt{N}}\right)^2 S^{\rm c}_\tau\left(\frac{\pi w}{\sqrt{N}}, \frac{\pi z}{\sqrt{N}}\right)\bigg|_{\tau=1-\alpha^2/N} = \frac{\pi}{2}\int_0^1 u\, {\rm e}^{-\alpha^2 u^2} \sin \pi u(w-z)\, {\rm d}u, \tag{6.28}$$

where on the lhs we have used the fact that the complex–complex correlations must be scaled by $(\pi/\sqrt{N})^2$ for each independent two-dimensional coordinate (x, y) to account for the measure in (6.24). Substituting this into (4.7) and noting too that

$${\rm e}^{(y_l^2 - x_l^2)/(1+\tau)} \,{\rm erfc}\left(\sqrt{\frac{2}{1-\tau^2}}\, y_l\right) \to {\rm erfc}\left(\frac{\pi y_l}{\alpha}\right)$$

gives the explicit weakly non-symmetric limiting form of $\rho^{\rm c}_{(n)}$.

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